Lecture 1: consider the Lie algebra $sl_2 = \mathbb{C}\cdot E \oplus \mathbb{C}\cdot F \oplus \mathbb{C}\cdot H$, $[E,F]=H$, $[H,E]=2E$, $[H,F]=-2F$

and its universal enveloping algebra

$$U(sl_2) = \mathbb{C}\langle E,F,H\rangle \big/ EF-FE=H,\ HE-EH=2E,\ HF-FH=-2F$$

They have the same representation theory, in particular the $n+1$ dimensional irreducible representation on which the generators act by

$$E = \begin{pmatrix} 0 & n & & & \\ & 0 & n-1 & & \\ & & 0 & \ddots & \\ & & & 0 & 2 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ & & & n-1 & 0 \\ & & & & n & 0 \end{pmatrix}, \quad H = \begin{pmatrix} n & & & & & \\ & n-2 & & & & \\ & & n-4 & & & \\ & & & \ddots & & \\ & & & & 4-n & \\ & & & & & 2-n \\ & & & & & & -n \end{pmatrix}$$

Def (Drinfeld-Jimbo quantized enveloping algebra, or "quantum group" for short)

$$U_q(sl_2) = \mathbb{C}(q)\langle e,f,k\rangle \Big/ ef-fe = \frac{k-k^{-1}}{q-q^{-1}},\ ke = q^2 ek,\ kf = q^{-2} fk,\ kk^{-1}=1.$$

(also makes sense when $q \in \mathbb{C}^*$, not a root of unity)

The meaning of this definition is that e is like E, f is like F, k is like q^H; in fact, the structure constants of $U_q(sl_2)$ are basically replacements of those of $U(sl_2)$, in the sense that any integer n is replaced by $\frac{q^n-q^{-n}}{q-q^{-1}} = [n]$

for instance, if you replace every $k \in \mathbb{Z}$ by $[k] \in \mathbb{Z}[q^{\pm 1}]$, the $n+1$ dimensional irrep of $U(sl_2)$ becomes a $n+1$ dimensional irrep of $U_q(sl_2)$.

Bialgebra structure: $U(sl_2)$: $\underbrace{\Delta(E) = 1\otimes E + E\otimes 1}_{\text{primitive}}$, $\Delta(F) = 1\otimes F + F\otimes 1$, $\Delta(H) = 1\otimes H + H\otimes 1$

$U_q(sl_2)$: $\Delta(e) = k\otimes e + e\otimes 1$, $\Delta(f) = 1\otimes f + f\otimes k^{-1}$, $\underbrace{\Delta(k) = k\otimes k}_{\text{group-like}}$

- Consider any bialgebra A, with coproduct denoted by $\Delta(a) = a_1 \otimes a_2$, $\Delta^{(2)}(a) = (\Delta\otimes Id)\circ\Delta(a) = (Id\otimes\Delta)\circ\Delta(a) = a_1\otimes a_2 \otimes a_3$ etc (exp(primitive) = group-like)

If A fails to be co-commutative (i.e. $\Delta \neq \Delta^{op}$), then the permutation does not give an isomorphism of A-modules $V\otimes W \not\cong W\otimes V$, for $V,W \in Rep(A)$

A acts on tensor product via $A \xrightarrow{\Delta} A\otimes A \to End(V\otimes W)$

But it may still be that non-trivial isomorphisms exist! For instance, suppose we had $\forall V,W \in Rep(A)$ isomorphisms

$\tilde{R}_{VW} : V \otimes W \xrightarrow{\sim} W \otimes V$ which satisfy the compatibility conditions

called R-matrices

① $$\begin{tikzcd} V\otimes W\otimes U \arrow[r, "Id_V\otimes \tilde{R}_{WU}"] \arrow[rr, bend right, "\tilde{R}_{V\otimes W, U}"'] & V\otimes U\otimes W \arrow[r, "\tilde{R}_{VU}\otimes Id_W"] & U\otimes V\otimes W \end{tikzcd}$$

② Yang-Baxter equation

$$\begin{tikzcd} & W\otimes V\otimes U \arrow[r, "Id_W\otimes \tilde{R}_{VU}"] & W\otimes U\otimes V \arrow[dr, "\tilde{R}_{WU}\otimes Id_V"] & \\ V\otimes W\otimes U \arrow[ur, "\tilde{R}_{VW}\otimes Id_U"] \arrow[dr, "Id_V\otimes \tilde{R}_{WU}"'] & & & U\otimes W\otimes V \\ & V\otimes U\otimes W \arrow[r, "\tilde{R}_{VU}\otimes Id_W"] & U\otimes V\otimes W \arrow[ur, "Id_U\otimes \tilde{R}_{VW}"'] & \end{tikzcd}$$

(when $A = U_q(sl_2)$, these properties lead to the construction of the Jones polynomial, as prescribed by Witten and Reshetikhin – Turaev)

Suppose you want the R-matrices $\tilde{R}_{VW}$ to come from a "universal R-matrix" $R \in A \otimes A$, i.e. $\forall\, V, W \in Rep(A)$, we have

$$R_{VW} : V\otimes W \to V \otimes W \text{ given by } act_V \otimes act_W (R)$$

and we set $\tilde{R}_{VW} = swap \circ R_{VW}$?

Then the fact that $\tilde{R}_{VW}$ is an intertwiner boils down to ⓪ $R \circ \Delta(a) = \Delta^{op}(a) \circ R$ $\forall\, a \in A$

while properties ① and ② boil down to

① $(\Delta \otimes Id_A)(R) = R_{13} R_{23}$ as identities in $A\otimes A\otimes A$, where $R_{12} = R \otimes Id_A$, $R_{23} = Id_A \otimes R$

② $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$

Which bialgebras possess universal R-matrices? The quasitriangular ones!

<u>Def</u> (Drinfeld): suppose a bialgebra A has sub-bialgebras A^+ and A^-, and a pairing

$$A^+ \otimes A^- \xrightarrow{\langle \cdot , \cdot \rangle} \text{ground field}, \text{ such that}$$

$\langle a, b'b'' \rangle = \langle \Delta(a), b' \otimes b'' \rangle$, $\forall\, a, a', a'' \in A^+$

$\langle a'a'', b \rangle = \langle a' \otimes a'', \Delta^{op}(b) \rangle$, $b, b', b'' \in A^-$

} A^+ and $A^{-,coop}$ are dual bialgebras

and we have $A = A^+ \otimes A^-$ and the relation $a_1 b_1 \langle a_2, b_2 \rangle = b_2 a_2 \langle a_1, b_1 \rangle$ $\forall\, a \in A^+, b \in A^-$

any product of elements from A^+ and A^- can be written uniquely with the A^+'s left of the A^-'s (where $\Delta(a) = a_1 \otimes a_2$, $\Delta(b) = b_1 \otimes b_2$)

Thm (Drinfeld) : if A is a quasitriangular bialgebra, finite-dimensional and the pairing $\langle \cdot, \cdot \rangle : A^+ \otimes A^- \to$ ground field is non-degenerate, then the canonical tensor of the pairing

$$R = \sum_{\substack{\text{dual bases} \\ \{a_i\} \in A^+, \{b^i\} \in A^-}} a_i \otimes b^i$$

is a universal R-matrix

check properties ⓪, ①, ② using the fact that

$\langle R, b \otimes - \rangle = b$

$\langle - \otimes a, R \rangle = a$

$\forall a \in A^+, b \in A^-$

It turns out that $A = U_q(sl_2)$ is quasitriangular with $A^+ = \mathbb{C}(q)\langle e, k \rangle$, $A^- = \mathbb{C}(q)\langle f, k^{-1} \rangle$

and the pairing is generated by $\begin{cases} \langle k, k^{-1} \rangle = q^2 \\ \langle e, f \rangle = \dfrac{1}{q^{-1} - q} \end{cases} \Rightarrow \langle e^k, f^k \rangle = \dfrac{q^{\frac{k(k-1)}{2}} [k]_q!}{(q^{-1} - q)^k}$

However, it is ~~neither~~ not finite-dimensional and there is some overlap between A^+ and A^- due to $\{k^{\mathbb{Z}}\}$

To fix this, one lets $q = \exp(\hbar)$ and works over $\mathbb{C}[[\hbar]]$

Def: $U_\hbar(sl_2) = \mathbb{C}[[\hbar]] \langle e, f, H \rangle \Big/ ef - fe = \dfrac{\exp(\hbar H) - \exp(-\hbar H)}{\exp(\hbar) - \exp(-\hbar)}, \; \begin{matrix} He - eH = 2e \\ Hf - fH = -2f \end{matrix}$

generated in $\hbar$-adic topology, i.e. infinite sums of the form $a_0 + \hbar a_1 + \hbar^2 a_2 + \hbar^3 a_3 + \dots$ are allowed

define $A^+ = \mathbb{C}[[\hbar]]\langle e \rangle$

$A^- = \mathbb{C}[[\hbar]]\langle f \rangle$

$A^0 = \mathbb{C}[[\hbar]]\langle H \rangle$

$A^{\geq} = \mathbb{C}[[\hbar]]\langle e, H \rangle$

$A^{\leq} = \mathbb{C}[[\hbar]]\langle f, H \rangle$

because $\langle aX, bY \rangle = \langle a, b \rangle \langle X, Y \rangle$ for all $a \in A^+$, $b \in A^-$, $X, Y \in A^0$, the canonical tensor of the pairing $A^{\geq} \otimes A^{\leq} \to \mathbb{C}[[\hbar]]$ splits up as the product of the canonical tensor of the restriction $A^+ \otimes A^- \to \mathbb{C}[[\hbar]]$ and a contribution coming from A^0

$$\Downarrow$$

a universal R-matrix for $U_\hbar(sl_2)$ is $\mathcal{R} = \exp\left(\frac{\hbar \cdot H \otimes H}{2}\right) \cdot \mathcal{R}'$

(proof in Kassel's Quantum Groups, Theorem XVII.4.2.

call this $\mathcal{R}' = \sum_{k=0}^{\infty} e^k \otimes f^k \cdot \dfrac{(q^{-1} - q)^k}{q^{\frac{k(k-1)}{2}} [k]_q!}$